Diffusion model of evolution of superthermal high-energy particles under scaling in the early Universe

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Abstract

The evolution of a superthermal relic component of matter is studied on the basis of non-equilibrium model of Universe and the Fokker-Planck type kinetic equation offered by one of the authors.

1 Introduction

The diffusion equation describing cosmological evolution of superthermal particles under the assumption of interactions scaling recovery in the range of superhigh energies was studied in paper [1] :

$$\frac{\partial \mathcal{G}}{\partial \tau} = \frac{1}{x^2}\frac{\partial}{\partial x}x^2\left(\frac{\partial \mathcal{G}}{\partial x} + 2b(\tau)\mathcal{G}\right), \tag{1}$$

where the dimensionless function $b(\tau)$ is determined as:

$$b(\tau) = \int\limits_0^\infty \mathcal{G}(\tau, x)x dx; \tag{2}$$

It is necessary to solve the equation (1) with initial and boundary conditions of the form:

$$\mathcal{G}(0, x) = G(x); \tag{3}$$

$$\lim_{x\to\infty} G(\tau, x)x^3 = 0, \tag{4}$$

moreover the function $G(x)$ must satisfy the integral conditions:

$$\int\limits_0^\infty G(x)x^2 dx = 1; \tag{5}$$

$$\int\limits_0^\infty G(x)x^3 dx = 1. \tag{6}$$

As the detailed researches have shown, the solution of kinetic equation (1) in case of $b(\tau) = 0$ obtained in [1] becomes correct only at greater times of evolution,

therefore it becomes problematical to establish parameters relationship of this solution with the initial distribution. In this paper we discuss the evolution of system at small times.

It is necessary to note that the exact solution of the kinetic equation in diffusive approximation (1), satisfying the normalization relations (5), (6), is an equilibrium ulatrarelativistic Bolzman distribution with conformal temperature $\tau = \frac{1}{3}$:

$$f_0 = \frac{27}{2} e^{-3x}. \tag{7}$$

2 Numerical model of the initial distribution

Let's consider the initial distribution analogous to Fermi-Dirac distribution, given in the form of infinitely differentiable and integrable function:

$$G_0(x) = \frac{A}{e^{\xi x - y} + 1}, \tag{8}$$

where A,ξ,y - parameters of the initial distribution. These parameters must be such that normalization relations (5),(6) fulfill automatically. Thus, we have two algebraic relations on three parameters, solving which through parameter y we find:

$$\xi(y) = \frac{\int\limits_0^\infty \frac{t^3 dt}{e^{(t-y)} + 1}}{\int\limits_0^\infty \frac{t^2 dt}{e^{(t-y)} + 1}}; \quad A(y) = \frac{\xi^3(y)}{\int\limits_0^\infty \frac{t^2 dt}{e^{(t-y)} + 1}} \tag{9}$$

Slow convergence of improper integrals in (9) leads to necessity of their transformation to following form that is more suitable for numeric calculations:

$$J_1(y) = \int\limits_0^\infty \frac{t dt}{e^{(t-y)} + 1} \equiv$$

$$y \int\limits_0^\infty \frac{dx}{e^x + 1} + \int\limits_0^\infty \frac{x dx}{e^x + 1} + y \int\limits_0^y \frac{dx}{e^{-x} + 1} - \int\limits_0^y \frac{x dx}{e^x + 1}. \tag{10}$$

$$J_2(y) = \int\limits_0^\infty \frac{t^2 dt}{e^{(t-y)} + 1} \equiv y^2 \int\limits_0^\infty \frac{dx}{e^x + 1} +$$

$$2y \int\limits_0^\infty \frac{x dx}{e^x + 1} + \int\limits_0^\infty \frac{x^2 dx}{e^x + 1} + y^2 \int\limits_0^y \frac{dx}{e^{-x} + 1} - 2y \int\limits_0^y \frac{x dx}{e^x + 1} + \int\limits_0^y \frac{x^2 dx}{e^x + 1}. \tag{11}$$

$$J_3(y) = \int\limits_0^\infty \frac{t^3 dt}{e^{(t-y)}+1} \equiv y^3 \int\limits_0^\infty \frac{dx}{e^x+1} + 3y^2 \int\limits_0^\infty \frac{xdx}{e^x+1}$$

$$+3y \int\limits_0^\infty \frac{x^2 dx}{e^x+1} + \int\limits_0^\infty \frac{x^3 dx}{e^x+1} + y^3 \int\limits_0^y \frac{dx}{e^{-x}+1} -$$

$$3y^2 \int\limits_0^y \frac{xdx}{e^x+1} + 3y \int\limits_0^y \frac{x^2 dx}{e^x+1} - \int\limits_0^y \frac{x^3 dx}{e^x+1}. \tag{12}$$

Using known representations of Riemann ζ - functions and Bernoulli numbers (e.g., see [6]):

$$\int\limits_0^\infty \frac{x^{n-1}}{e^x+1} dx = (1-2^{1-n})\Gamma(n)\zeta(n); \tag{13}$$

$$\int\limits_0^\infty \frac{x^{2n-1}}{e^x+1} dx = \frac{2^{2n-1}-1}{2n} \pi^{2n} B_n \tag{14}$$

and introducing dimensionless functions:

$$S(n,y) = \int\limits_0^y \frac{x^n}{e^{-x}+1} dx, \quad (n = 0, 1, 2, ...), \tag{15}$$

$(S(0,y) = \ln(1+e^y)/2)$ we get expressions for (10)-(12):

$$J_1(y) = y \ln 2 + \frac{\pi^2}{12} + yS(0,y) - S(1,y); \tag{16}$$

$$J_2(y) = y^2 \ln 2 + y\frac{\pi^2}{6} + \frac{3}{2}\zeta(3) + y^2 S(0,y) -$$

$$2yS(1,y) + S(2,y); \tag{17}$$

$$J_3(y) = y^3 \ln 2 + y^2 \frac{\pi^2}{4} + y\frac{9}{2}\zeta(3) + \frac{7}{120}\pi^4 +$$

$$y^3 S(0,y) - 3y^2 S(1,y) + 3yS(2,y) - S(3,y). \tag{18}$$

Thus, by means of introduced functions (10)-(12) we find expression for $b(\tau)$:

$$b_0(y) = A(y) \int\limits_0^\infty \frac{xdx}{e^{\xi(y)x-y}+1} \equiv$$

$$\frac{A(y)}{\xi^2(y)} \int\limits_0^\infty \frac{xdx}{e^{x-y}+1} = \frac{J_3(y)J_1(y)}{J_2^2(y)}. \tag{19}$$

Presentation of (9) by means of $S(n, y)$ and Riemann ζ - functions makes numeric computations simpler. The results of integration are shown on Figure 1,2 and we see that $\xi(y)$ is monotone increasing, $A(y)$ is monotone decreasing function.

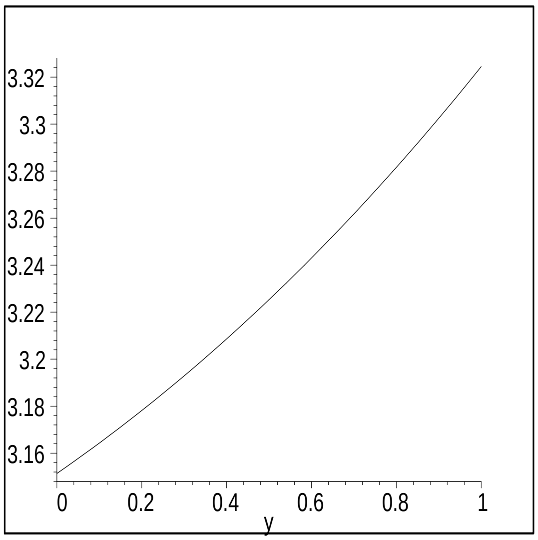

Figure 1: *Plot of* $\xi(y)$ *function.*

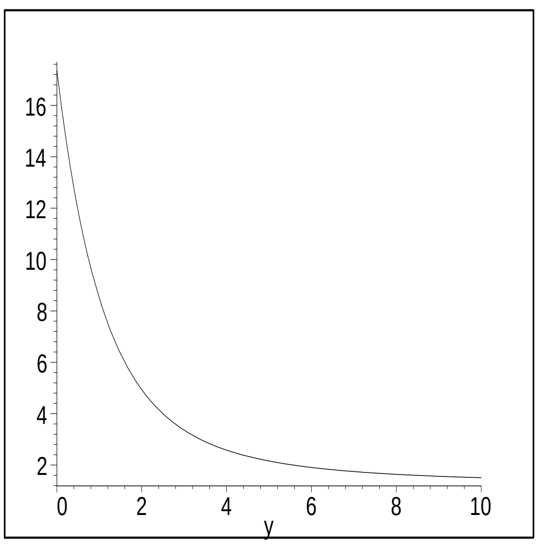

Figure 2: *Plot of* $A(y)$ *function.*

As a result, normalized initial distribution function is defined by one arbitrary parameter, y:

$$G_0(x,y) = \frac{A(y)}{e^{\xi(y)x-y}+1}, \tag{20}$$

which controls the the degree of non-equilibrium of initial distribution (8), Figure 3.

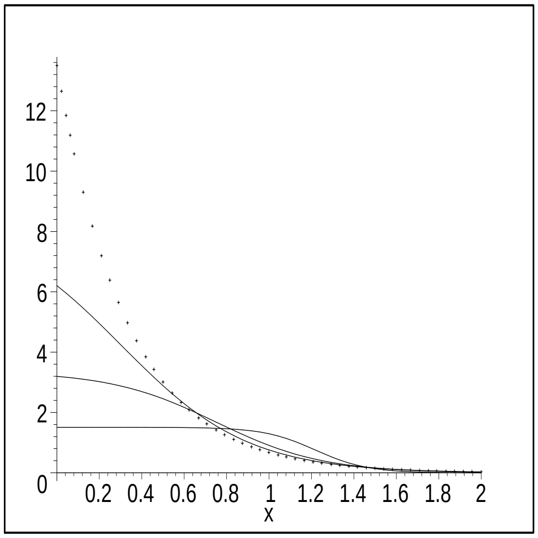

Figure 3: *Normalized initial distribution $G_0(x,y)$ according to parameter y: top-down: $y=1$, $y=3$, $y=10$. The initial distribution (8) is shown by dotted line.*

3 Diffusion equation and integral conservation laws

As known (see, e.g. [5]), the strict consequences from general relativistic kinetic theory in case of elastic collisions are integral conservation laws of particles total number and their energy. The solution of diffusion equation (1), obtained on the basis of general relativistic kinetic equations, also must satisfy these laws. Therefore it is necessary to verify this fact.

Multiplying by x^2 the both sides of equation (1) and integrating received expression in parts on all interval of impulsive variable x, we get:

$$\frac{d}{d\tau}\int\limits_0^\infty x^2 G(x,\tau)dx = \left[x^2\left(\frac{\partial G}{\partial x} + 2b(\tau)G\right)\right]\Bigg|_0^\infty. \tag{21}$$

Assuming later on:

$$\lim_{x\to 0} x^2 G; \quad \lim_{x\to 0} x^2 \frac{\partial G}{\partial x} = 0; \tag{22}$$

$$\lim_{x\to \infty} x^2 G = 0; \quad \lim_{x\to \infty} x^2 \frac{\partial G}{\partial x} = 0; \tag{23}$$

we obtain from (21):

$$\int\limits_0^\infty x^2 G(x,\tau)dx = \text{Const.} \tag{24}$$

According to (5) the constant at the right side of (24) equals to 1.

Multiplying now by x^3 the both sides of (1) and integrating received expression in parts on all interval, we get:

$$\frac{d}{d\tau}\int\limits_0^\infty x^3 G(x,\tau)dx = \left[x^3\left(\frac{\partial G}{\partial x} + 2b(\tau)G\right)\right]\Bigg|_0^\infty -$$

$$-\int\limits_0^\infty x^2 \left(\frac{\partial G}{\partial x} + 2b(\tau)G\right)dx \tag{25}$$

Considering (22), (24), (4) and once more integrating received expression in parts, we find:

$$\frac{d}{d\tau}\int\limits_0^\infty x^3 G(x,\tau)dx = -\left. x^2 G\right|_0^\infty +$$

$$+2\int\limits_0^\infty xGdx - 2b(\tau)\int\limits_0^\infty x^2 Gdx. \tag{26}$$

Taking into account relations (2),(5), and also (22),(24), we finally obtain from (26):

$$\int\limits_0^\infty x^3 G(x,\tau)dx = \text{Const.} \tag{27}$$

According to (6) the constant at the right side of (27) equals to 1.

It is enough in order to $G(x)$ function's degree of magnitude satisfy the following strong inequalities for realization of relations (22),(24),(4):

$$G(x)|_{x\to 0} < \frac{1}{x}, \quad G(x)|_{x\to\infty} < \frac{1}{x^3}. \tag{28}$$

We also note that realization of the energy conservation law is provided with the presence of term with $b(\tau)$ coefficient at the right side of diffusion equation (1). If this term is missing, the process of energy transmitting at small times is not considered.

4 Expansion of diffusion equation by smallness of τ

This approximation match up the early stages of universe evolution, when particle interactions are inessential:

$$\tau \ll 1. \tag{29}$$

Then the term at the left side of diffusion equation is main and taking into account the initial distribution (3) we have:

$$\frac{\partial G}{\partial \tau} = 0, \Rightarrow G(x,\tau) = G_0(x). \tag{30}$$

Expanding the right side of equation (1), substituting expression (30) as the initial distribution and integrating in time variable we obtain the first correction. Sequentially iterating this procedure we get the recurring formula for finding higher approximations:[1]

$$G_{k+1} = \int \frac{1}{x^2}\frac{\partial}{\partial x} x^2 \left[\frac{\partial G_k}{\partial x} + 2\sum_{i=0}^{k}\left(G_i \int_0^\infty G_{k-i} x dx\right)\right] d\tau, \tag{31}$$

Considering that correction of k-order is proportional to τ^k and integrating (31), we get:

$$G_{k+1} = \frac{\tau^{k+1}}{k+1}\frac{1}{x^2}\frac{\partial}{\partial x} x^2 \left[\frac{\partial g_k}{\partial x} + 2\sum_{i=0}^{k}\left(g_i \int_0^\infty g_{k-i} x dx\right)\right]. \tag{32}$$

Differentiating in expression (32), we finally get:

$$G_{k+1} = \frac{\tau^{k+1}}{k+1}\left\{\frac{\partial^2 g_k}{\partial x^2} + \frac{2}{x}\frac{\partial g_k}{\partial x} + 2\sum_{i=0}^{k}\left[\left(\frac{\partial g_i}{\partial x} + \frac{2 g_i}{x}\right) b_{k-i}\right]\right\}. \tag{33}$$

We also find the recurring formula for determining function $b(\tau)$ in equation (1). Considering (32) we obtain according to (2):

$$b_{k+1} = \frac{\tau^{k+1}}{k+1}\int_0^\infty \frac{dx}{x}\frac{\partial}{\partial x} x^2 \left[\frac{\partial g_k}{\partial x} + \right.$$

[1] Here and further, if not mentioned especially, $k \in \mathbb{N}$.

$$2\sum_{i=0}^{k}\left(g_i\int_0^\infty g_{k-i}xdx\right)\Bigg]. \tag{34}$$

Integrating in parts an integral in (34) and supposing henceforward:

$$\lim_{x\to 0} xg_i = 0; \quad \lim_{x\to 0} x\frac{\partial g_k}{\partial x} = 0; \tag{35}$$

$$\lim_{x\to \infty} xg_i = 0; \quad \lim_{x\to \infty} x\frac{\partial g_k}{\partial x} = 0. \tag{36}$$

we get finally:

$$b_{k+1} =$$

$$\frac{\tau^{k+1}}{k+1}\int_0^\infty \left[\frac{\partial g_k}{\partial x} + 2\sum_{i=0}^{k}(g_i b_{k-i})\right]dx. \tag{37}$$

Now we prove that the corrections to normalized initial distribution (8) can not change the total number density and energy density at every step of iterations.

Multiplying by x^2 the both sides of (32) and integrating on all interval, we get:

$$\int_0^\infty x^2 G_{k+1}dx = \frac{\tau^{k+1}}{k+1}\Bigg\{x^2\Bigg[\frac{\partial g_k}{\partial x}+$$

$$2\sum_{i=0}^{k}\left(g_i\int_0^\infty g_{k-i}xdx\right)\Bigg]\Bigg\}\Bigg|_0^\infty. \tag{38}$$

Supposing henceforward:

$$\lim_{x\to 0} x^2 g_i = 0; \quad \lim_{x\to 0} x^2\frac{\partial g_k}{\partial x} = 0; \tag{39}$$

$$\lim_{x\to \infty} x^2 g_i = 0; \quad \lim_{x\to \infty} x^2\frac{\partial g_k}{\partial x} = 0. \tag{40}$$

we obtain from (38):

$$\int_0^\infty x^2 G_{k+1}dx = 0. \tag{41}$$

Multiplying by x^3 the both sides of (32) and integrating on all interval, we get:

$$\int_0^\infty x^3 G_{k+1}dx =$$

$$= \frac{\tau^{k+1}}{k+1}\left\{x^3\left[\frac{\partial g_k}{\partial x} + 2\sum_{i=0}^{k}\left(g_i\int_0^\infty g_{k-i}xdx\right)\right]\right\}\Bigg|_0^\infty -$$

$$\frac{\tau^{k+1}}{k+1}\int\limits_0^\infty x^2\left[\frac{\partial g_k}{\partial x}+2\sum_{i=0}^{k}\left(g_i\int_0^\infty g_{k-i}xdx\right)\right]dx. \quad (42)$$

Considering (39), (40), (27) and once more integrating received expression in parts, we find from (42):

$$\int\limits_0^\infty x^3 G_{k+1}dx=\frac{\tau^{k+1}}{k+1}\left\{-\left.x^2 g_k\right|_0^\infty+2\int\limits_0^\infty xg_k dx-\right.$$

$$\left.-2\int\limits_0^\infty x^2\sum_{i=0}^{k}\left(g_i\int_0^\infty g_{k-i}xdx\right)dx\right\}. \quad (43)$$

After an obvious simplifications

$$\int\limits_0^\infty x^3 G_{k+1}dx=\frac{\tau^{k+1}}{k+1}\left\{-\left.x^2 g_k\right|_0^\infty+2\int\limits_0^\infty xg_k dx-\right.$$

$$-2\int\limits_0^\infty xg_k dx\int\limits_0^\infty x^2 g_0 dx-$$

$$\left.-2\sum_{i=1}^{k}\left(\int\limits_0^\infty g_{k-i}xdx\int_0^\infty x^2 g_i dx\right)\right\}, \quad (44)$$

and taking into account expressions (5), (39), (40), (41), we get from (44) at last:

$$\int\limits_0^\infty x^3 G_{k+1}dx=0. \quad (45)$$

Thus, we certain that the distribution function iterations of each step don't change the total number of particles and energy, this is useful tool for calculation correctness. It follows from (45) that the correction of any order is alternating-sign on the interval $[0,+\infty)$. Hence, small time τ approximation is completely equivalent to expansion of exact function $G(x,\tau)$ to Taylor series by τ powers.

4.1 The first order approximation

As a first approximation according to recurring formula (32) we have:

$$G_1=\tau g_1; \quad (46)$$

$$g_1=\frac{1}{x^2}\frac{\partial}{\partial x}x^2\left(\frac{\partial G_0}{\partial x}+2b_0 G_0\right). \quad (47)$$

Substituting expression (47) into (2) and integrating in parts, also considering that function $G_0(x,y)$, defined by expression (8), with its derivatives approach to 0 at $x \to \infty$ faster than any of exponential functions and at $x \to 0$ has finite derivatives, we obtain:

$$b_1 = \tau \int_0^\infty \frac{dx}{x} \frac{\partial}{\partial x} \left(\frac{\partial G_0}{\partial x} + 2b_0 G_0(x) \right) =$$

$$= \tau \left[x \left(\frac{\partial G_0}{\partial x} + 2b_0 G_0(x) \right) \right] \Bigg|_0^\infty +$$

$$+\tau \int_0^\infty \left(\frac{\partial G_0}{\partial x} + 2b_0 G_0(x) \right) dx =$$

$$= -\tau G_0(0) + \tau 2b_0 \int_0^\infty G_0 dx \Rightarrow$$

$$b_1(\tau) = -\tau \frac{A(y)e^y}{e^y+1} + \tau 2b_0 \int_0^\infty G_0 dx.$$

Substituting here the initial distribution (8), we find:

$$\int_0^\infty G_0 dx = \frac{A(y)}{a(y)} \ln(1+e^y), \tag{48}$$

therefore

$$b_1(\tau) = \tau A(y) \left[\frac{2b_0 \ln(1+e^y)}{a(y)} - \frac{e^y}{e^y+1} \right]. \tag{49}$$

Then, substituting the function $G_0(x,y)$ into (47), we determine an explicit form of linear approximation to the initial distribution:

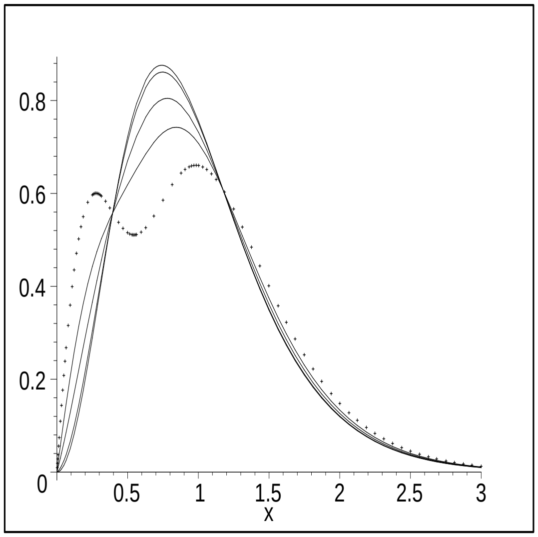

Figure 4: *Evolution of number density of particles,* $dn(\tau, x, y) = x^2 G(\tau, x, y)$, *as a first approximation at* $y = 1$. *Solid lines from left to right:* $\tau = 0; 0,01; 0,05; 0,1$, *dotted line-*$\tau = 0,2$.

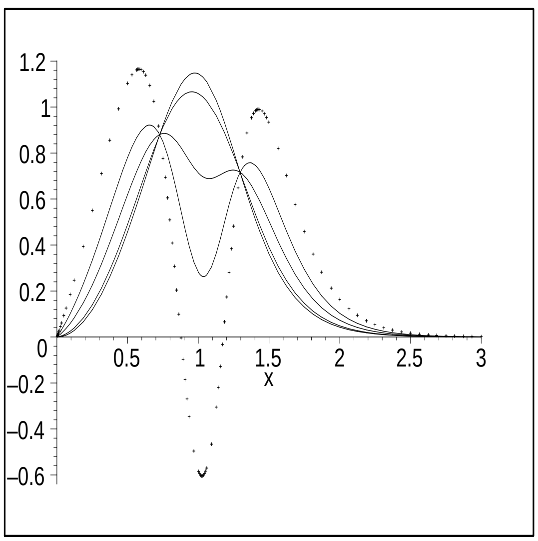

Figure 5: *Evolution of number density of particles,* $dn(\tau, x, y) = x^2 G(\tau, x, y)$, *as a first approximation at* $y = 6$. *Solid lines from left to right:* $\tau = 0; 0,01; 0,05; 0,1$, *dotted line-*$\tau = 0,2$.

$$g_1(x, y) = \frac{2A(y)}{x} \frac{e^{\xi(y)x-y}[2b_0(y) - \xi(y)] + 2b_0(y)}{(e^{\xi(y)x-y} + 1)^2} -$$

$$-\frac{A(y)\xi(y)[\xi(y)+2b_0(y)]e^{\xi(y)x-y}}{(e^{\xi(y)x-y}+1)^2}+$$

$$+\frac{2A(y)\xi(y)^2e^{2(\xi(y)x-y)}}{(e^{\xi(y)x-y}+1)^3}. \tag{50}$$

On the figures 4-11 is shown the evolution of distribution of number density and their energy density as a first approximation at the values of $y = 1, 3, 6, 10$, when the significant particles part of the initial distribution lies in the area of small energy values.

As we see from these and also following figures:

1. In number density and their energy density distributions always appear 2 maximums and 1 minimum, which shifts to area of higher energy values with increasing of y parameter;

2. Always appear such a moment of time, at which the minimum of distribution function takes negative value;

3. After a time the first minimum shifts to the area of lower energies, the second minimum - to the area of higher energies.

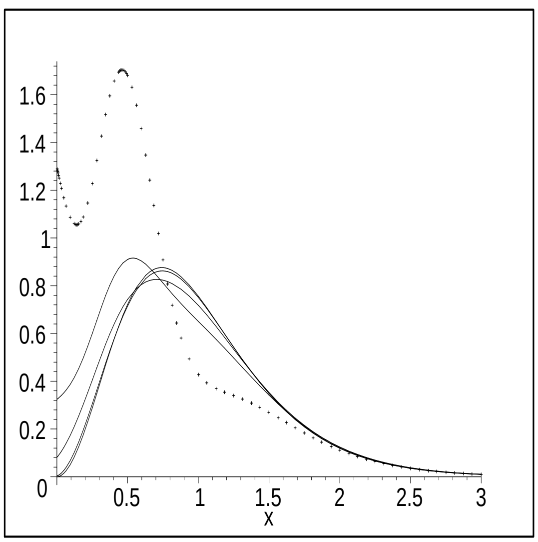

Figure 6: *Evolution of number density of particles,* $dn(\tau, x, y) = x^2G(\tau, x, y)$, *as a second approximation at* $y = 1$. *Solid lines from left to right:* $\tau = 0; 0,01; 0,05; 0,1$, *dotted line-*$\tau = 0,2$.

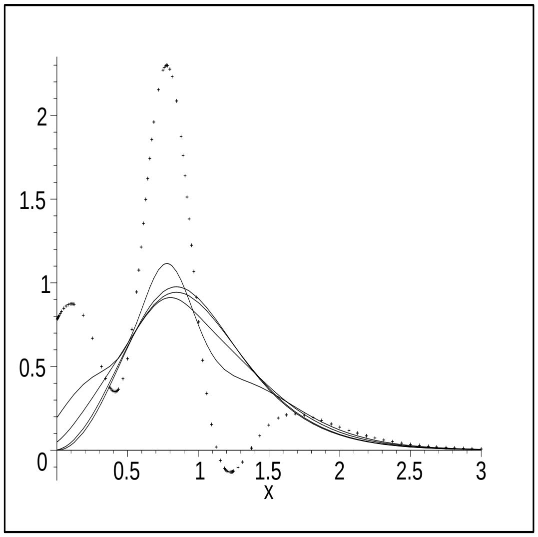

Figure 7: *Evolution of number density of particles,* $dn(\tau, x, y) = x^2 G(\tau, x, y)$, *as a second approximation at* $y = 3$*. Solid lines from left to right:* $\tau = 0; 0,01; 0,05; 0,1$, *dotted line-*$\tau = 0,2$.

4.2 The second order approximation

As a second approximation according to (32) we have:

$$G_2 = \frac{\tau^2}{2} g_2; \tag{51}$$

$$g_2 = \frac{1}{x^2} \frac{\partial}{\partial x} x^2 \left(\frac{\partial g_1}{\partial x} + 2g_1 b_0 + 2G_0 b_1 \right). \tag{52}$$

Substituting then G_0 and g_1 into (52), we find the explicit form of the second order approximation.[2]

[2] In case that the explicit form of the second order approximation is an unwieldy we don't cite it here

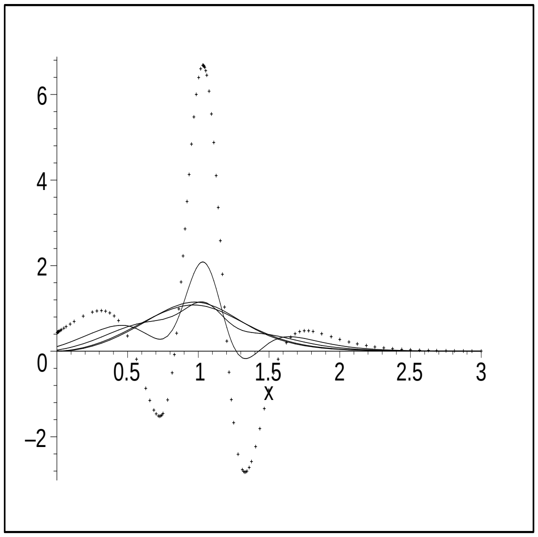

Figure 8: *Evolution of number density of particles,* $dn(\tau, x, y) = x^2 G(\tau, x, y)$*,*
as a second approximation at $y = 6$*. Solid lines from left to right:*
$\tau = 0; 0,01; 0,05; 0,1$*, dotted line-*$\tau = 0,2$*.*

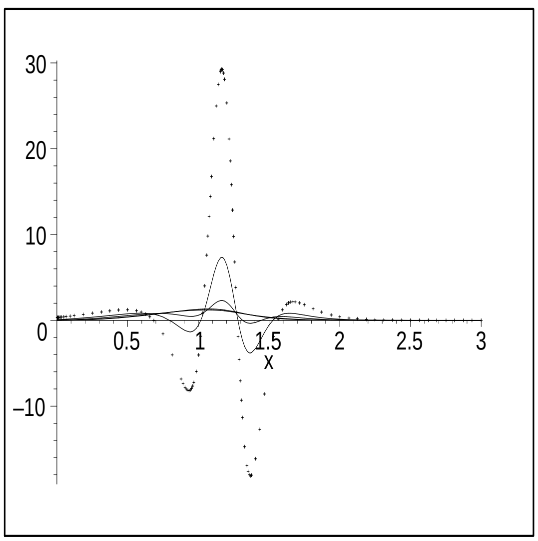

Figure 9: *Evolution of number density of particles,* $dn(\tau, x, y) = x^2 G(\tau, x, y)$*,*
as a second approximation at $y = 10$*. Solid lines from left to right:*
$\tau = 0; 0,01; 0,05; 0,1$*, dotted line-*$\tau = 0,2$*.*

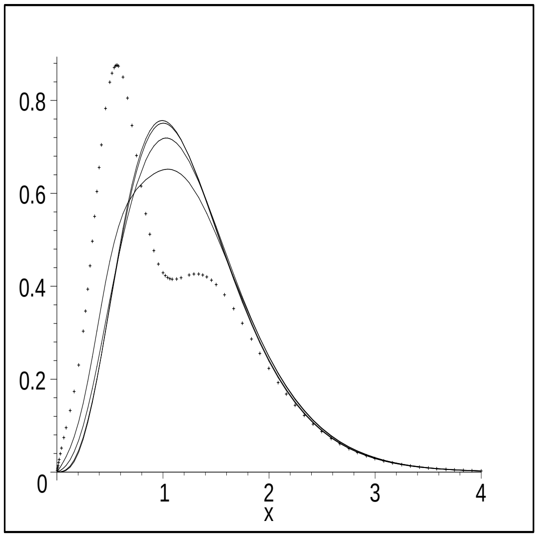

Figure 10: *Evolution of the energy distribution of particles,* $d\varepsilon(\tau, x, y) = x^3 G(\tau, x, y)$*, as a second approximation at* $y = 1$*. Solid lines from left to right:* $\tau = 0; 0,01; 0,05; 0,1$*, dotted line-*$\tau = 0,2$.

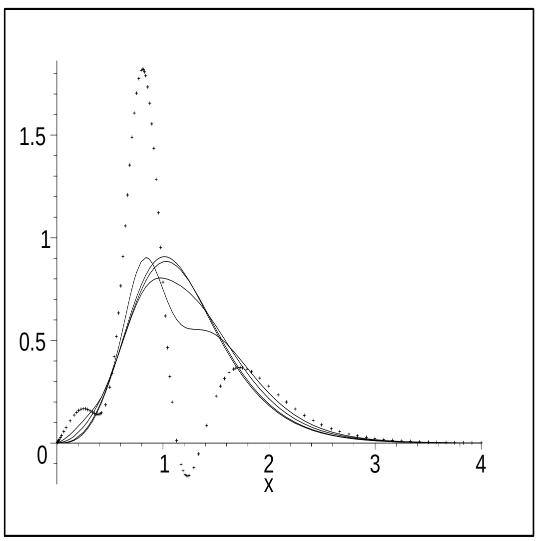

Figure 11: *Evolution of the energy distribution of particles,* $d\varepsilon(\tau, x, y) = x^3 G(\tau, x, y)$*, as a second approximation at* $y = 3$*. Solid lines from left to right:* $\tau = 0; 0,01; 0,05; 0,1$*, dotted line-*$\tau = 0,2$.

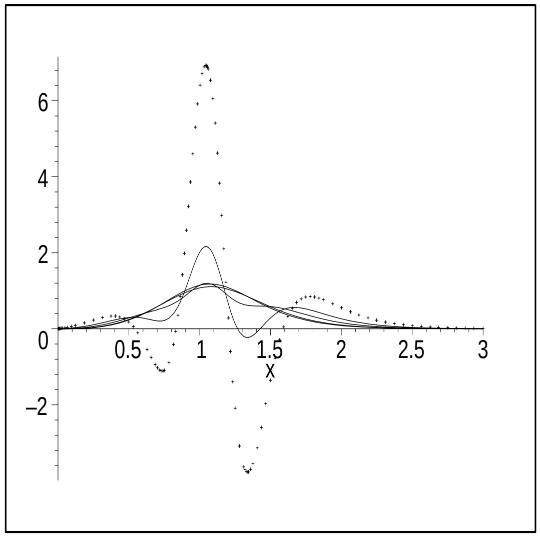

Figure 12: *Evolution of the energy distribution of particles,* $d\varepsilon(\tau, x, y) = x^3 G(\tau, x, y)$*, as a second approximation at* $y = 6$*. Solid lines from left to right:* $\tau = 0; 0,01; 0,05; 0,1$*, dotted line-*$\tau = 0,2$*.*

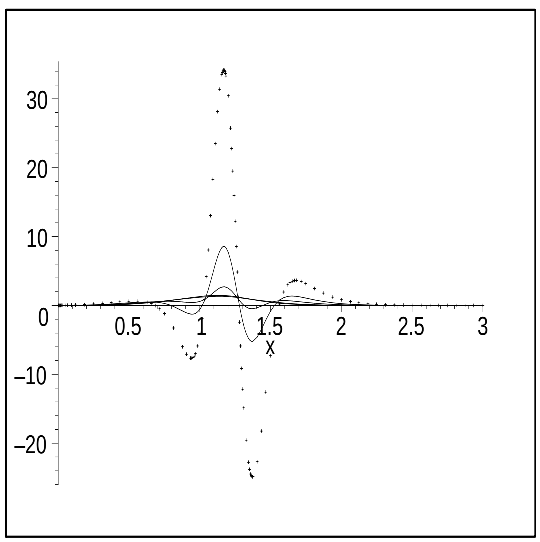

Figure 13: *Evolution of the energy distribution of particles,* $d\varepsilon(\tau, x, y) = x^3 G(\tau, x, y)$*, as a second approximation at* $y = 10$*. Solid lines from left to right:* $\tau = 0; 0,01; 0,05; 0,1$*, dotted line-*$\tau = 0,2$*.*

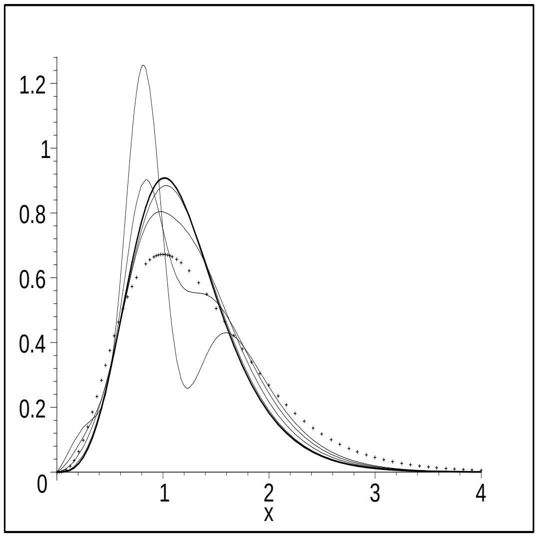

Figure 14: *Evolution of the energy distribution of particles, $d\varepsilon(\tau, x, y) = x^3 G(\tau, x, y)$, as a second approximation at $y = 3$. Solid lines from left to right: $\tau = 0; 0,01; 0,05; 0,1$, dotted line-$\tau = 0,2$.*

5 Conclusion

Since particles distribution function is nonnegative by definition, it is clear that penetration of distribution minimum to the negative values area is an implication of corrections smallness conditions failure in the area of concrete energy values. Examined approximations sufficiently describe the global properties of superthermal particles distribution. The fact of occurrence of two maximums in superthermal particles distribution is very significant. As a point of view offered in [3], [4] superhigh energy particles origin model, the first of these maximums at a later time can evolve to equilibrium distribution what certify our calculations, the second one can give us high-enegry tail of superthermal relic particles.

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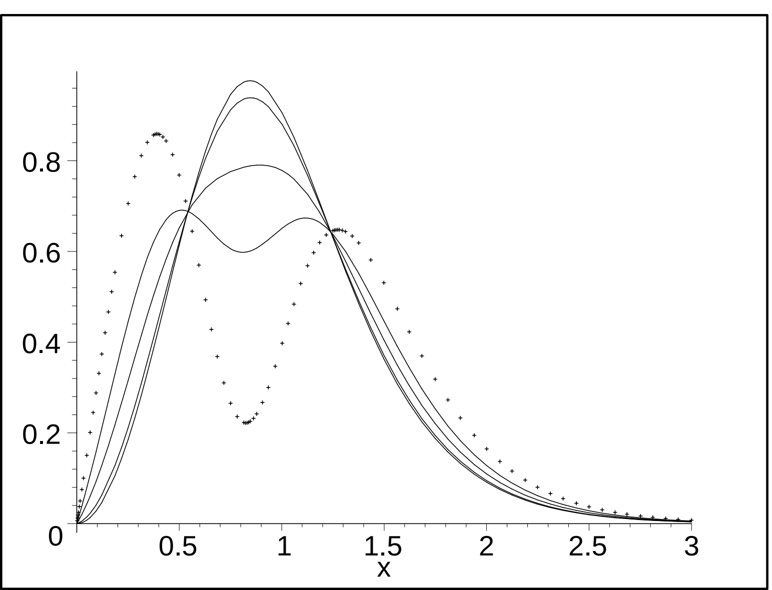

0.8
0.6
0.4
0.2
0
0.5
1
1.5
2
2.5
3
x